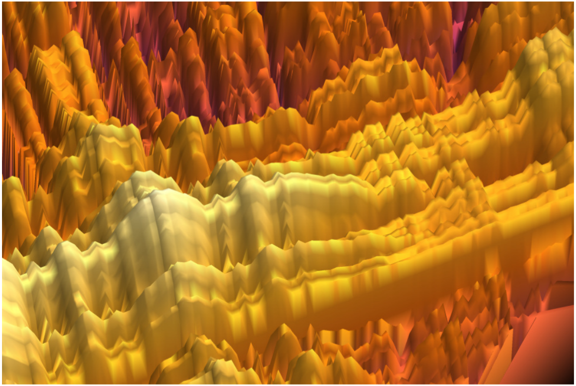

An introduction to core entropy

Giulio Tiozzo
University of Toronto

Summary

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Then the topological entropy of f is

$$h_{top}(f) := \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n+1}(\mathcal{U}))$$

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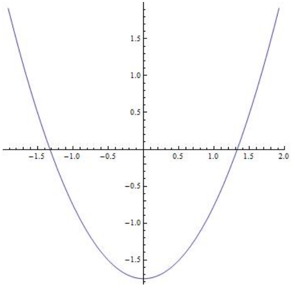

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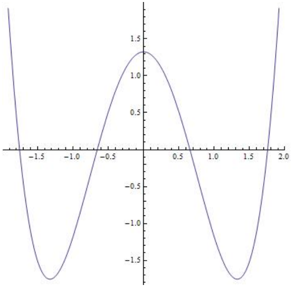

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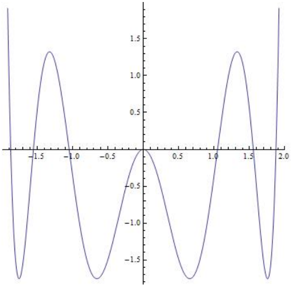

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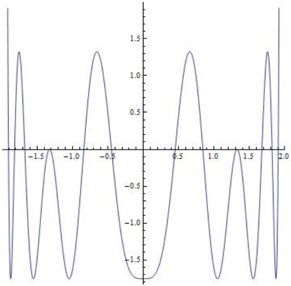

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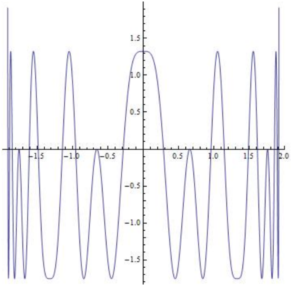

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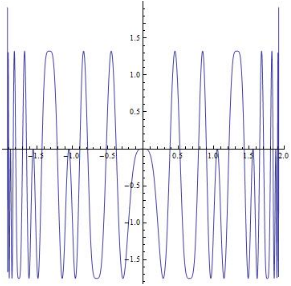

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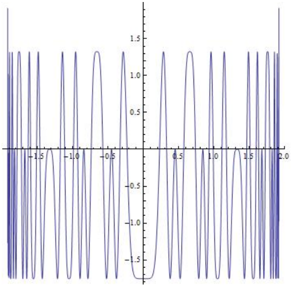

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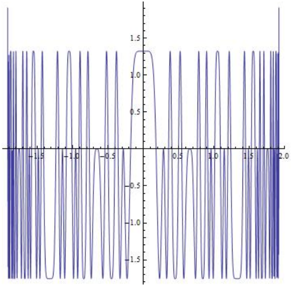

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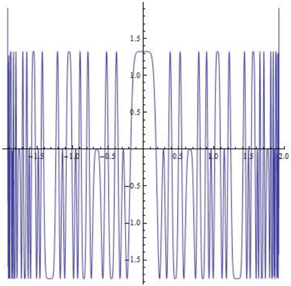

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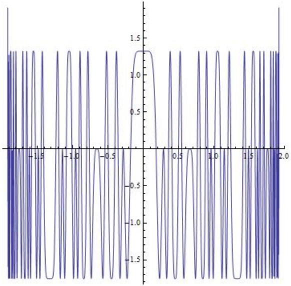

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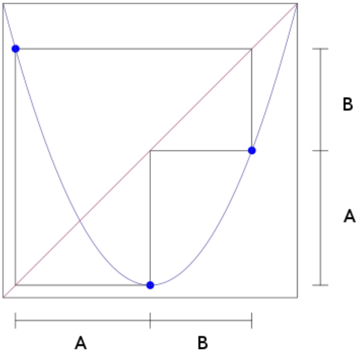

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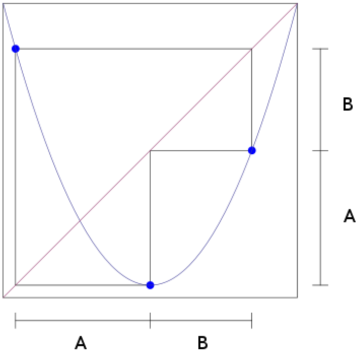

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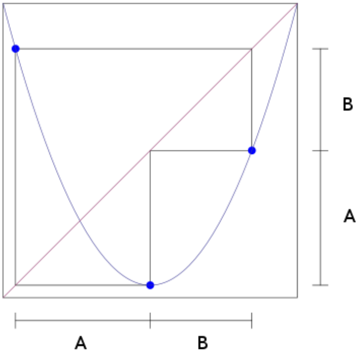

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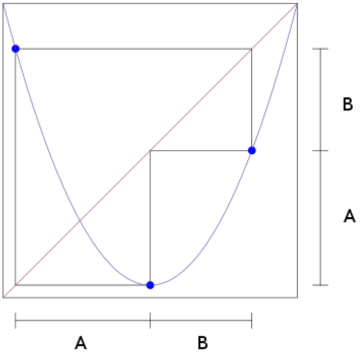

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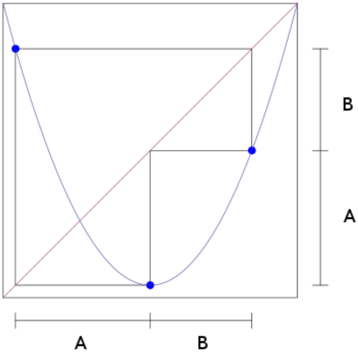

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Consider the real quadratic family

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How does entropy change with the parameter c?

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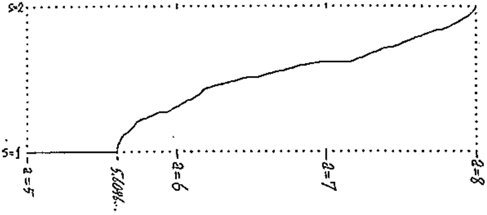

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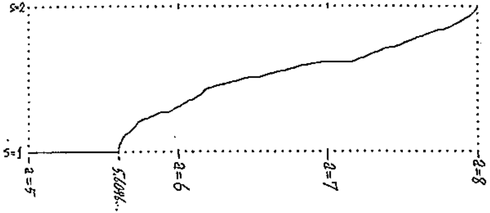

[Picture is for $f_a(x) = ax(1 - x)$.]

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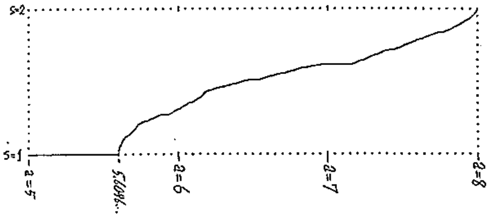

Question : Can we extend this theory to complex polynomials?

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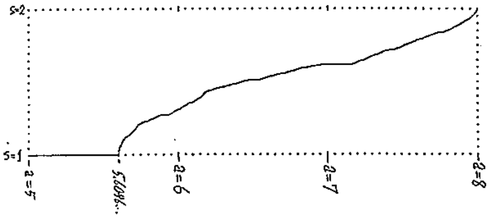
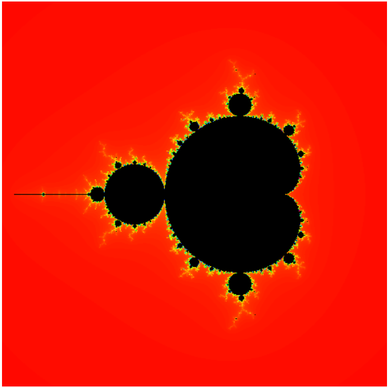


Remark. If we consider $f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ entropy is constant $h_{top}(f_c, \hat{\mathbb{C}}) = \log 2$. (Lyubich 1980)

Mandelbrot set

The Mandelbrot set $\mathcal{M}$ is the connectedness locus of the quadratic family

$$\mathcal{M} = \{c \in \mathbb{C} \ : \ f_c^n(0) \nrightarrow \infty\}$$



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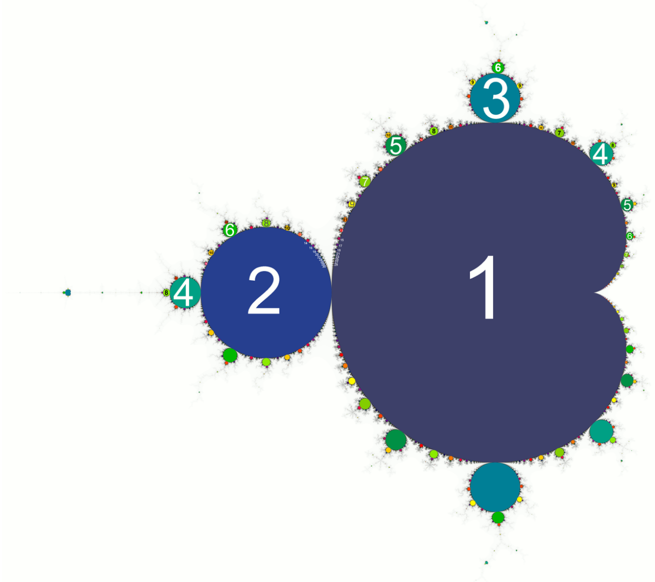

External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

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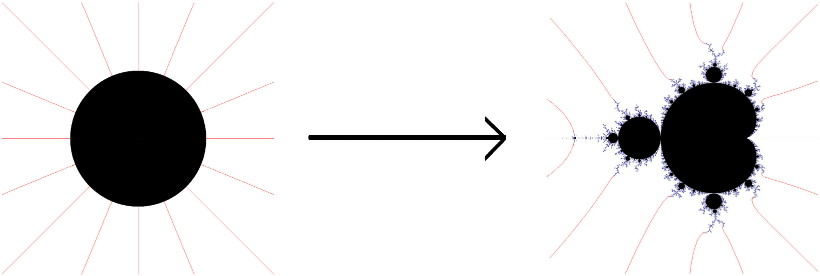

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The images of radial arcs in the disk are called external rays. Every angle $\theta \in \mathbb{R}/\mathbb{Z}$ determines an external ray

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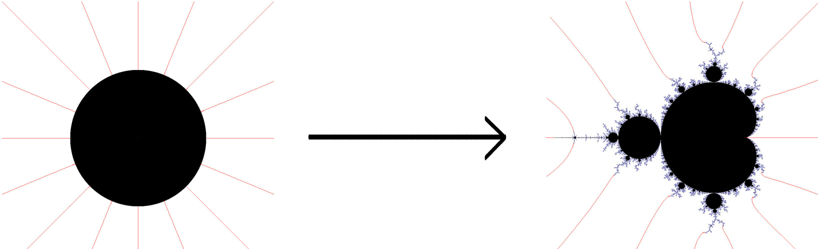

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As a consequence, the Mandelbrot set is homeomorphic to a quotient of the closed disk

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All rays land, and the boundary map $\mathbb{R}/\mathbb{Z} \to \partial\mathcal{M}$ is continuous.

As a consequence, the Mandelbrot set is homeomorphic to a quotient of the closed disk (hence locally connected).

Julia sets

Let $f_c(z) = z^2 + c$. Then the filled Julia set of f_c is the set of points which do not escape to infinity under forward iteration:

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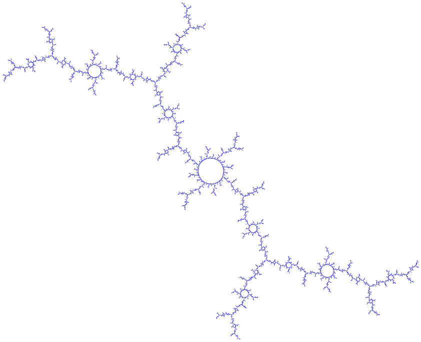

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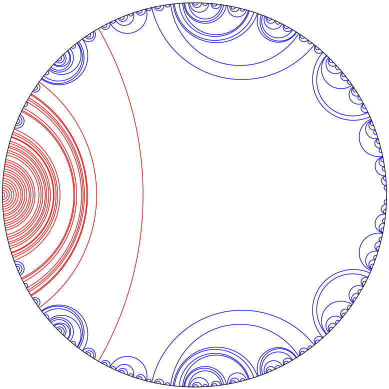
Thurston's quadratic minor lamination (QML)



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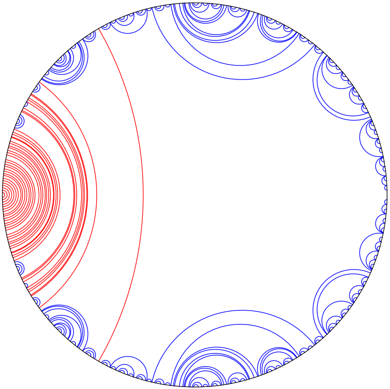

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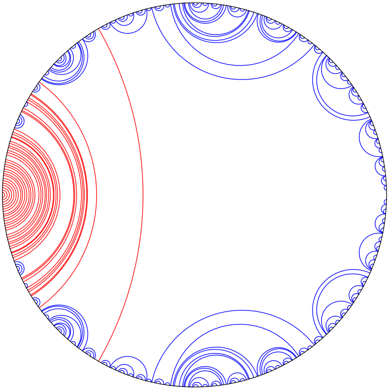

For each f_c, pick the minor leaf of the lamination for f_c (i.e, the ray pair landing at the critical value (or its root)). The QML is the union of all minor leaves for all $c \in \mathcal{M}$. The quotient $\mathcal{M}_{abs}$ of the disk by the lamination is a (locally connected) model for the Mandelbrot set, and homeomorphic to it if MLC holds.

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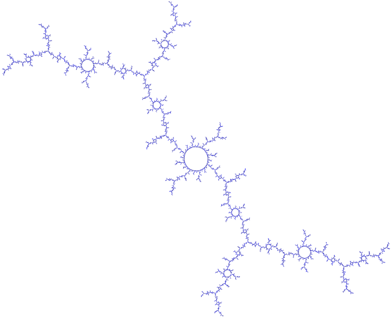

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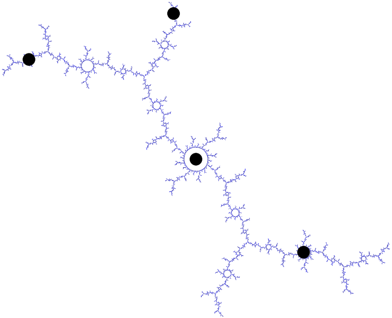

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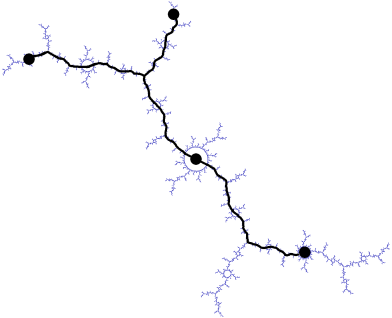

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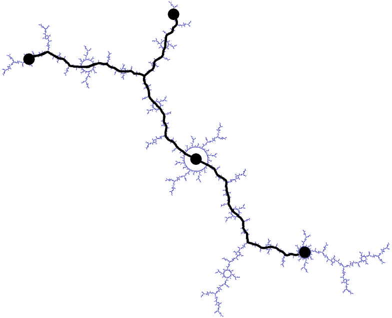

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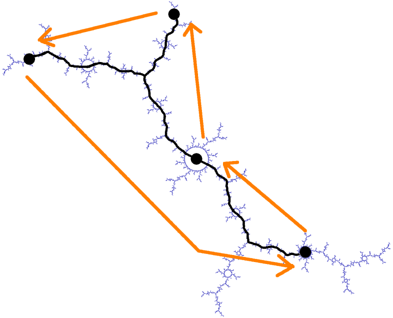

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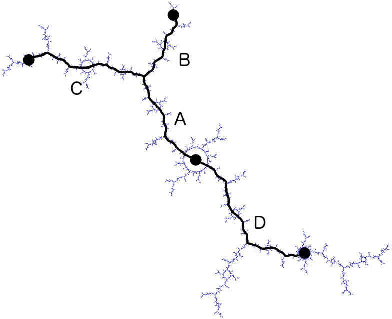

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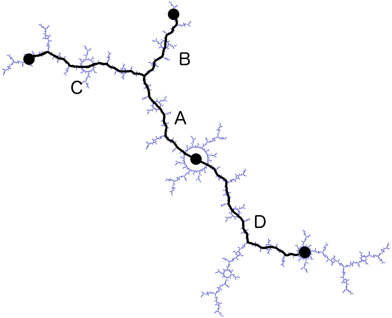

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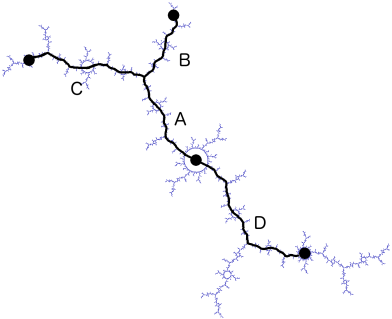

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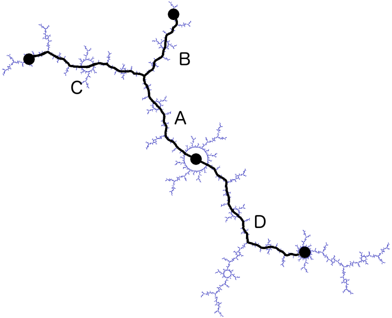

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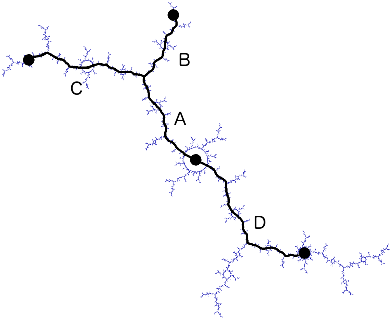

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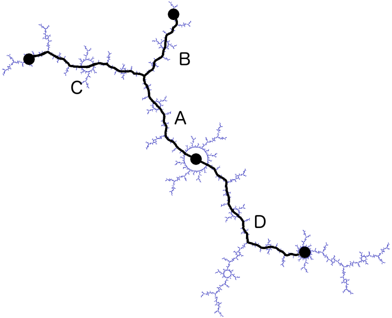

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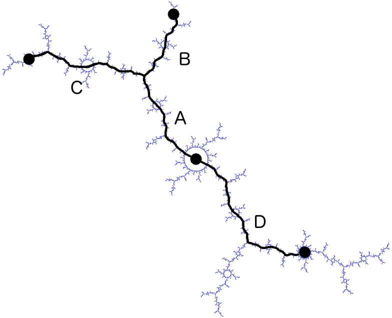

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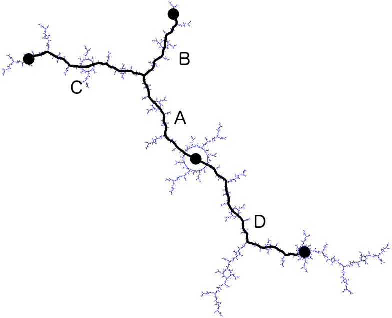

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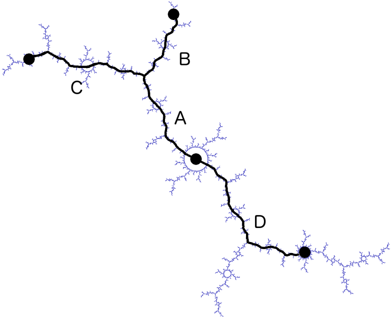

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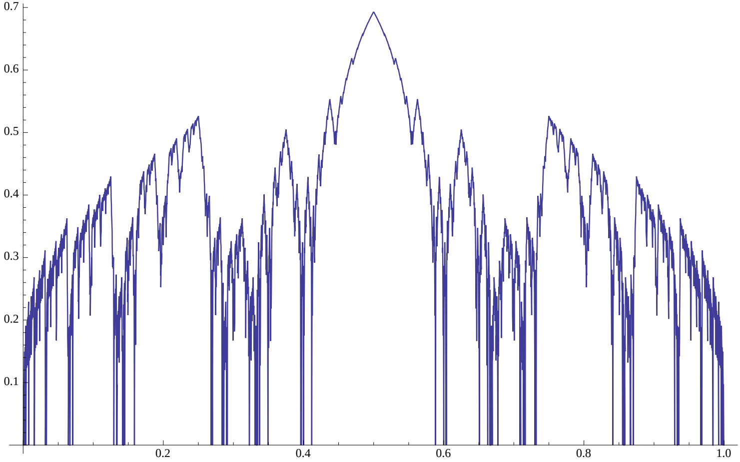
Question: How does $h(\theta)$ vary with the parameter θ?

Core entropy as a function of external angle (W. Thurston)



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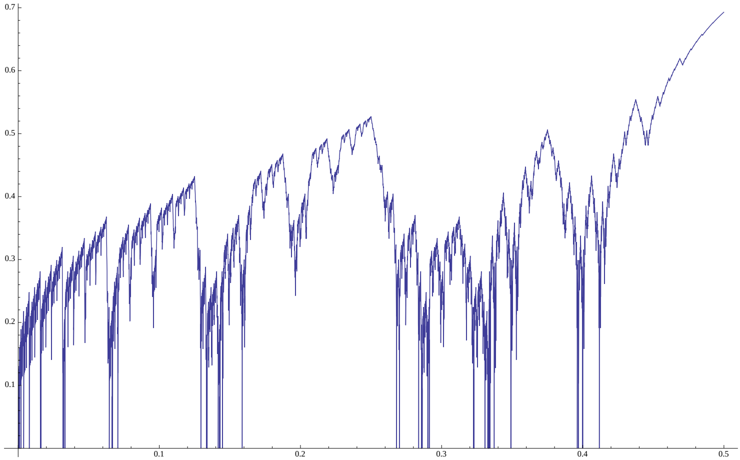

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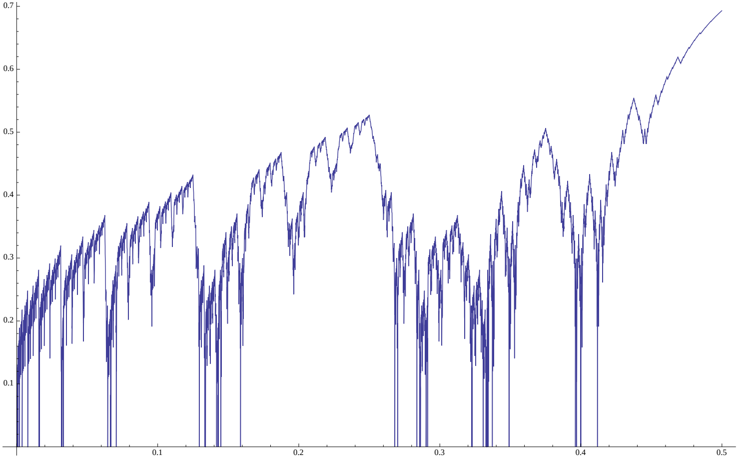

Question Can you see the Mandelbrot set in this picture?

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Theorem (Li Tao; Penrose; Tan Lei; Zeng Jinsong)

If $\theta_1 <_M \theta_2$, then

$$h(\theta_1) \leq h(\theta_2)$$

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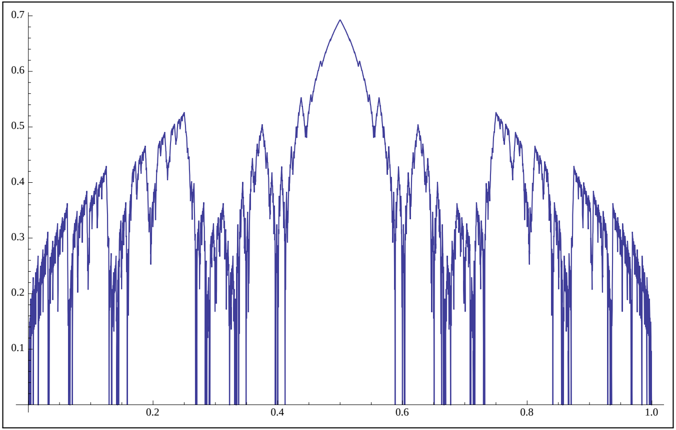
Theorem (T., Bruin-Schleicher)

If the Hubbard tree of f_c is topologically finite, then

$$\text{H. dim } B_c = \frac{h(f_c)}{\log 2}$$

The core entropy as a function of external angle

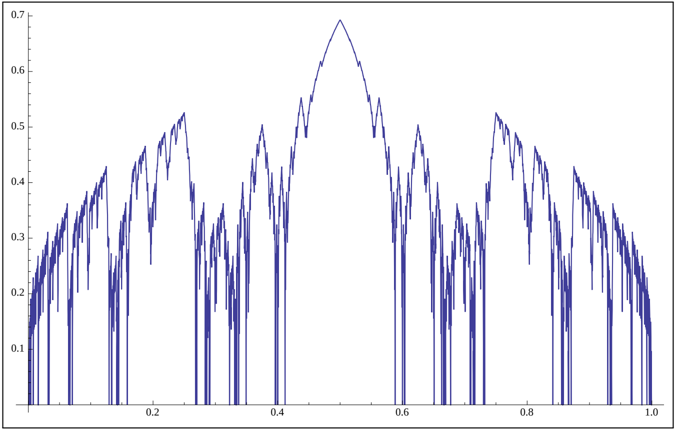
Question (Thurston, Hubbard):
Is $h(\theta)$ a continuous function of θ?



The Main Theorem: Continuity

Theorem (T.)

The core entropy function $h(\theta)$ extends to a continuous function from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.



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Denote $c_i := f^i(0)$ the i^{th} iterate of the critical point, and let

$$P := \{(c_i, c_j) \ \ i, j \geq 0\}$$

the set of pairs of postcritical points

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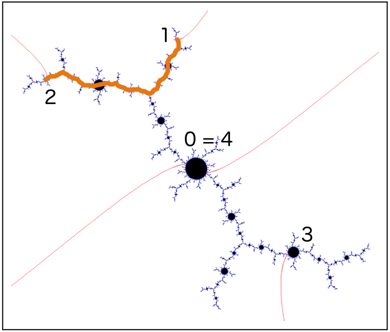

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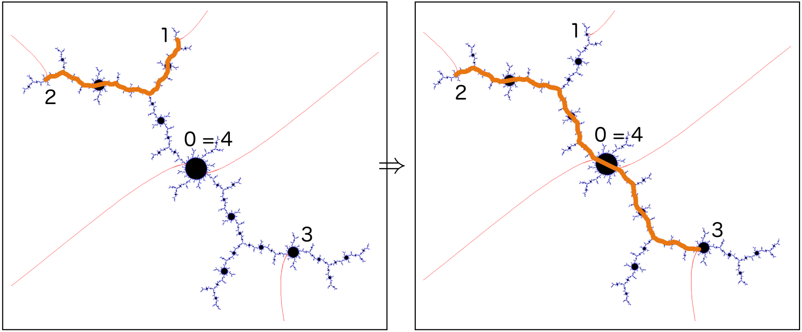

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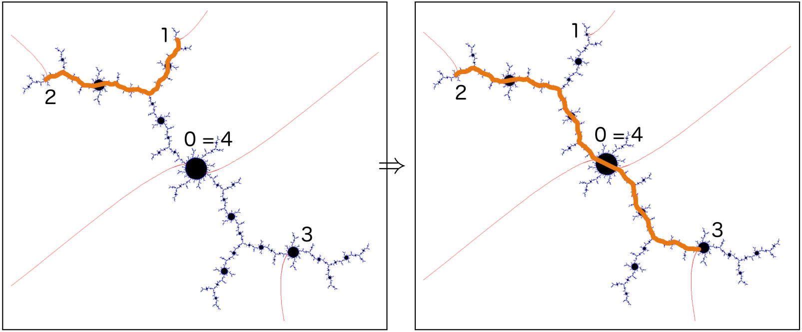

$(1, 2) \quad \Rightarrow \quad (2, 3)$

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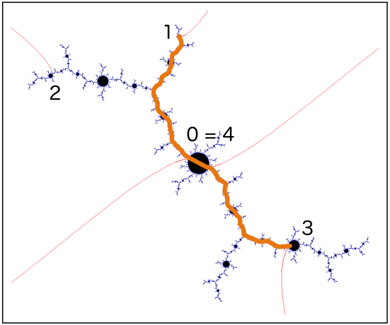

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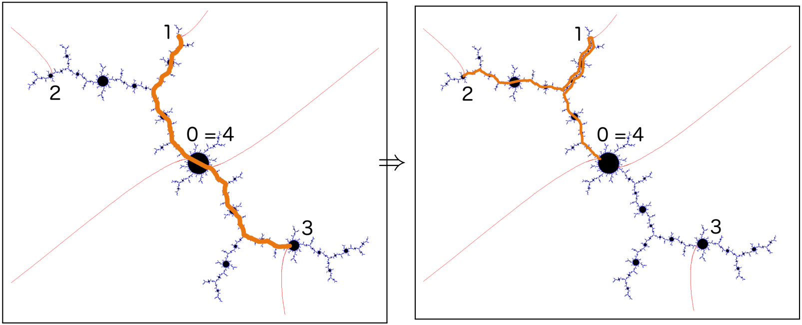

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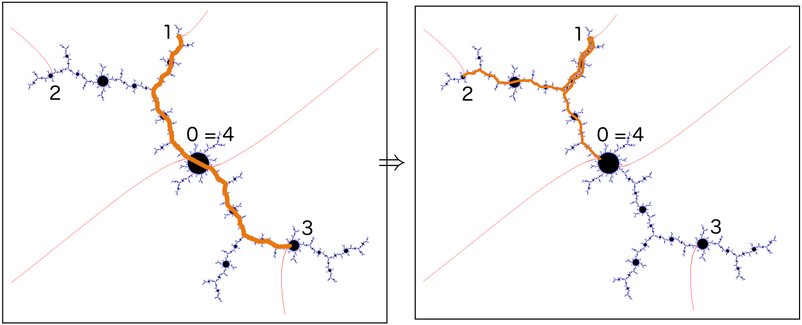

$$(1, 3) \quad \Rightarrow \quad (1, 2) + (1, 4)$$

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Theorem (Thurston; Tan Lei)

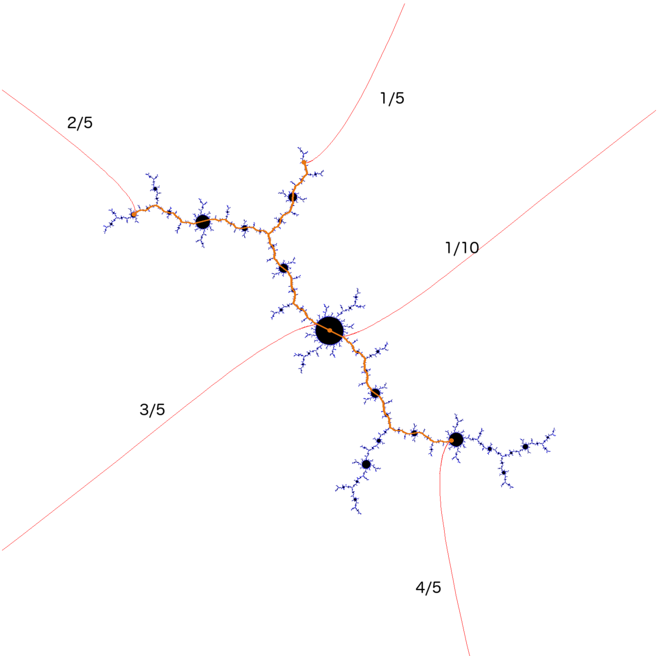
The entropy of f_θ is given by

$$h(\theta) = \log \lambda$$

where λ is the leading eigenvalue of A.

See also Gao, Jung.

The algorithm



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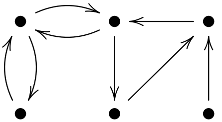

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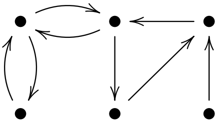

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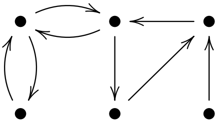

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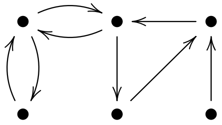

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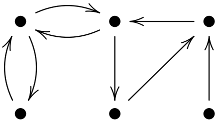

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$$P(t) = 1 - 2t^2 - t^3 + t^5$$

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Then we define the growth rate of Γ as :

$$r(\Gamma) := \limsup \sqrt[n]{C(\Gamma, n)}$$

where $C(\Gamma, n)$ is the number of closed paths of length n.

Computing entropy: the infinite clique polynomial

Let Γ with bounded outgoing degree and bounded cycles.

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Theorem

Let $\sigma \leq 1$. Then $P(t)$ defines a holomorphic function in the unit disk, and its root of minimum modulus is r^{-1}.

How to compute the core entropy without knowing complex dynamics

Let $\theta \in \mathbb{R}/\mathbb{Z}$, and $\theta_i := 2^{i-1}\theta \mod 1$, and consider the diameter $\{\theta/2, (\theta+1)/2\}$ (= major leaf).

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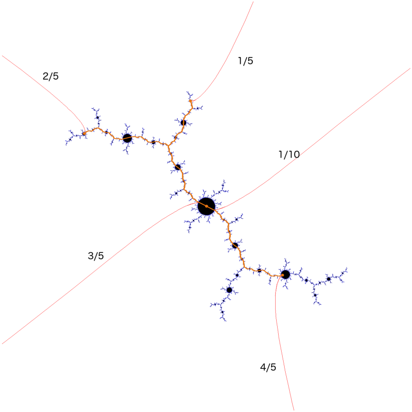

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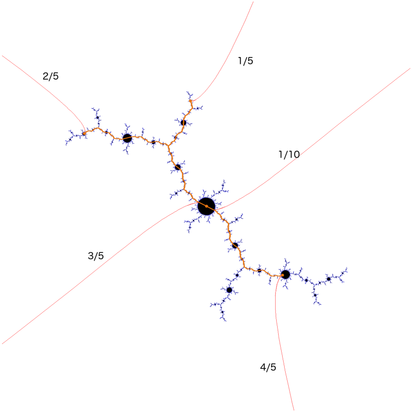

- (i, j) separated $\Leftrightarrow$ θ_i and θ_j lie on opposite side of diameter
- (i, j) non-separated $\Leftrightarrow$ θ_i and θ_j lie on same side of diam.

Wedges

				$\cdots$
			$(4,5)$	$\cdots$
		$(3,4)$	$(3,5)$	$\cdots$
	$(2,3)$	$(2,4)$	$(2,5)$	$\cdots$
$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$\cdots$

Labeled wedges

Label all pairs as either separated or non-separated

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	$\boxed{(2,3)}$	$(2,4)$	$\cdots$
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(The boxed pairs are the separated ones.)

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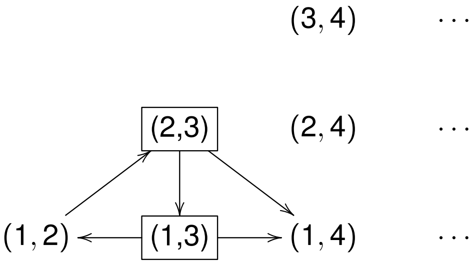

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and	$r(\theta_n) \to r(\theta)$	(growth rates)

Regularity properties of the core entropy

In fact:

Theorem (T. '15)

The core entropy is locally Hölder continuous at θ if $h(\theta) > 0$, and not locally Hölder at θ where $h(\theta) = 0$.

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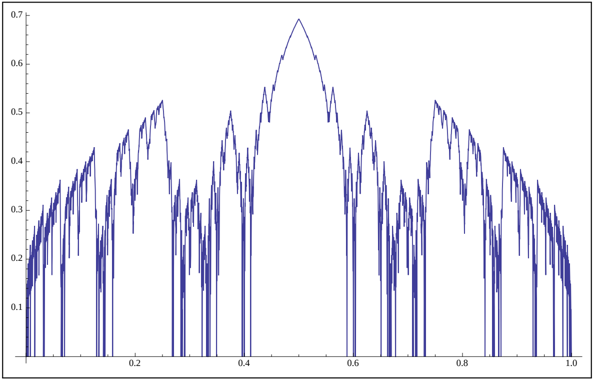

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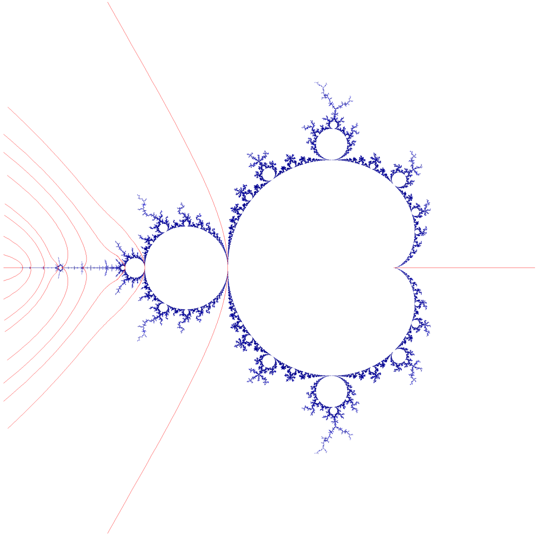
(Conjectured Isola-Politi, 1990)

Rays landing on the real slice of the Mandelbrot set



Harmonic measure

Given a subset A of $\partial\mathcal{M}$, the harmonic measure $\nu_{\mathcal{M}}$ is the probability that a random ray lands on A:

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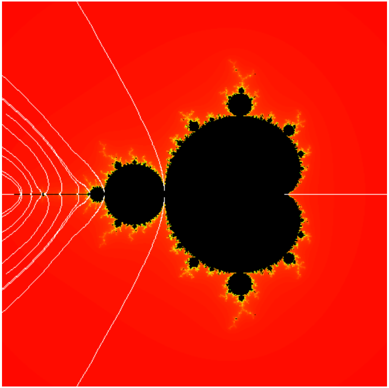
For instance, take $A = \mathcal{M} \cap \mathbb{R}$ the real section of the Mandelbrot set. How common is it for a ray to land on the real axis?



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The harmonic measure of the real axis is 0.

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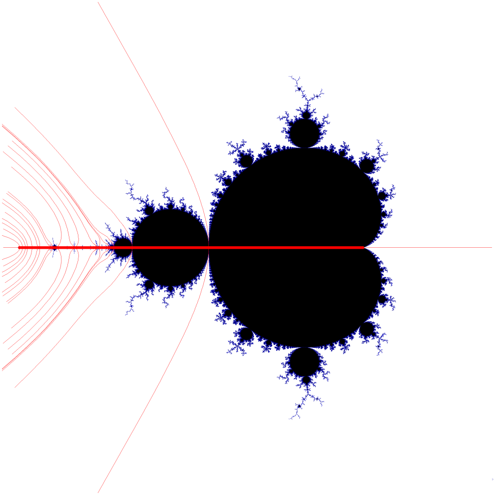

Sectioning $\mathcal{M}$

Given $c \in [-2, 1/4]$, we can consider the set of external rays which land on the real axis to the right of c:

$$P_c := \{\theta \in S^1 : R(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$

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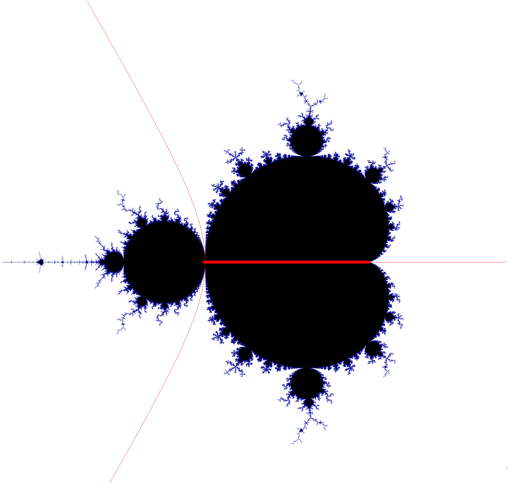

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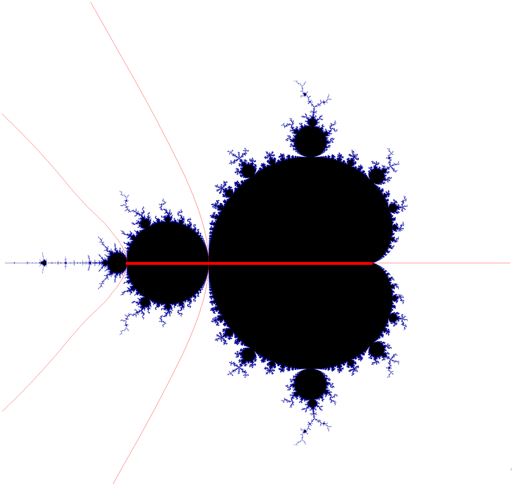

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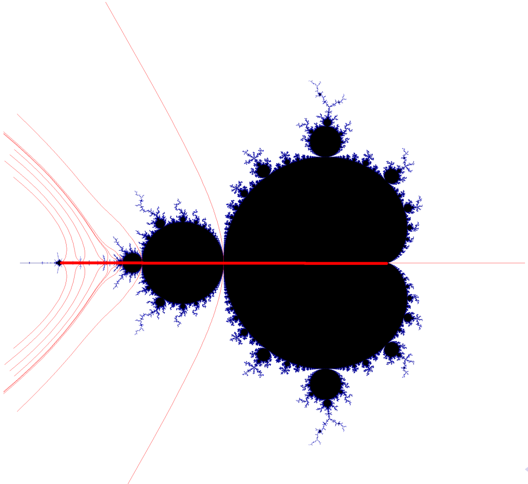

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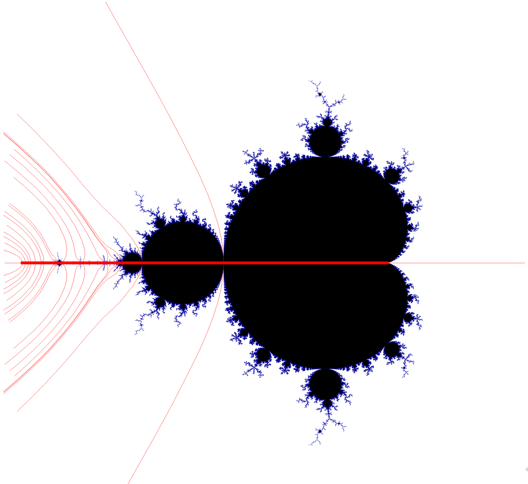

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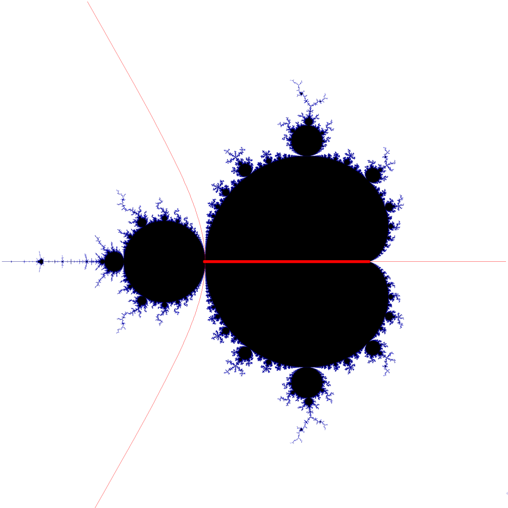
The function

$$c \mapsto \text{H.dim } P_c$$

decreases with c, taking values between 0 and 1.



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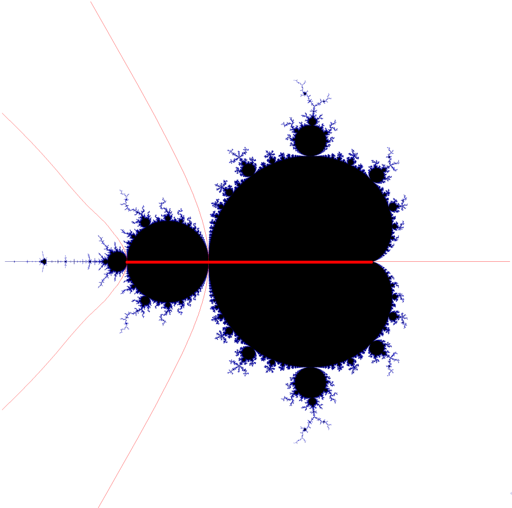

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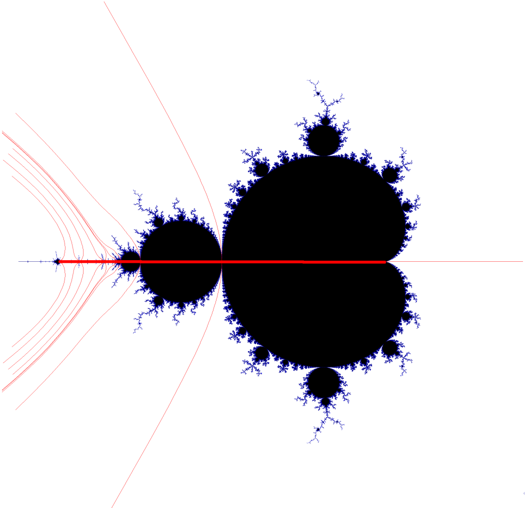

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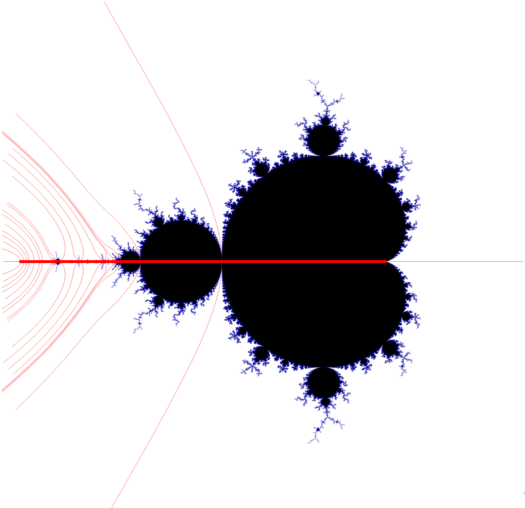

Entropy formula, real case

Theorem (T.)

Let $c \in [-2, 1/4]$*. Then*

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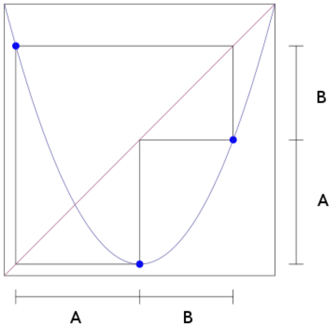

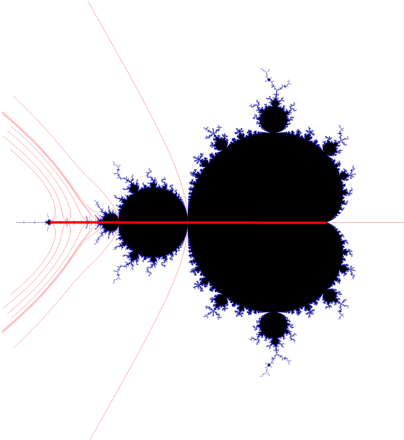

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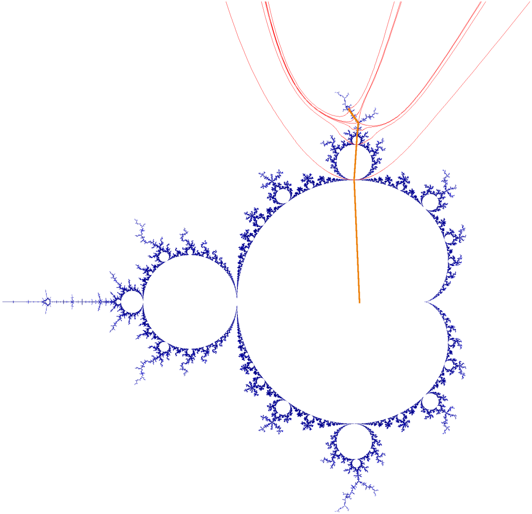
- It does not depend on MLC.
- It can be generalized to non-real veins.

Entropy formula along complex veins

A vein is an embedded arc in the Mandelbrot set.



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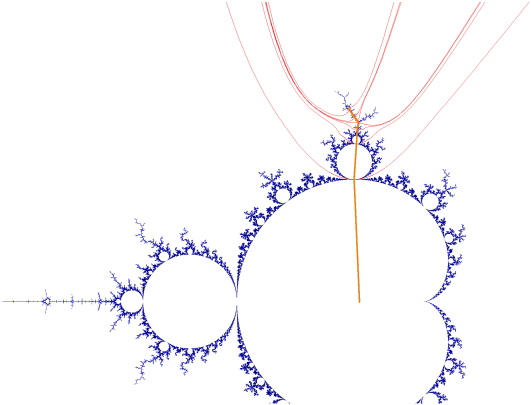

Given a parameter c along a vein, we can look at the set P_c of parameter rays which land on the vein between 0 and c.

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Theorem (T.; Jung)

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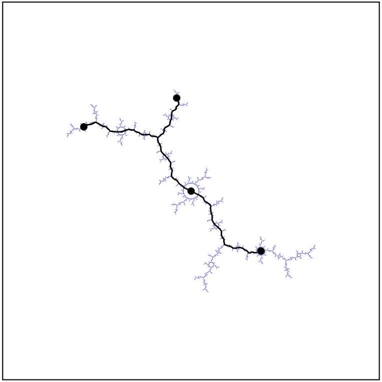

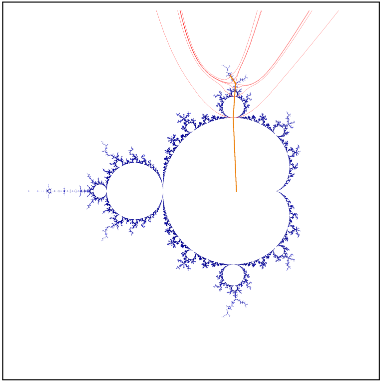

Further directions / questions

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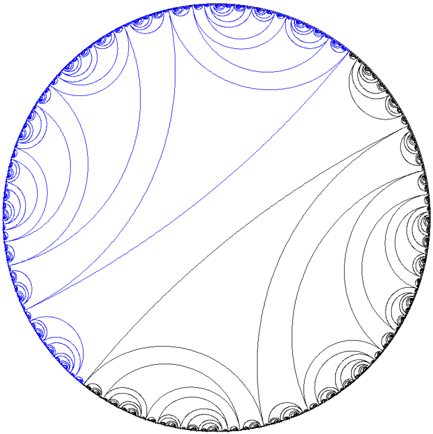

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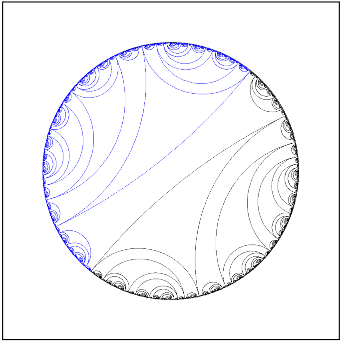

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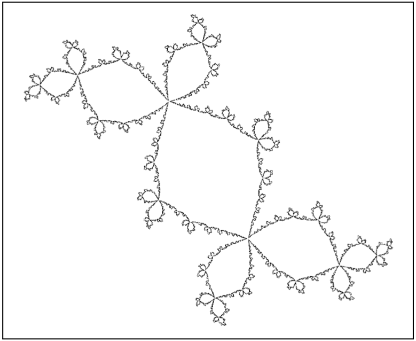

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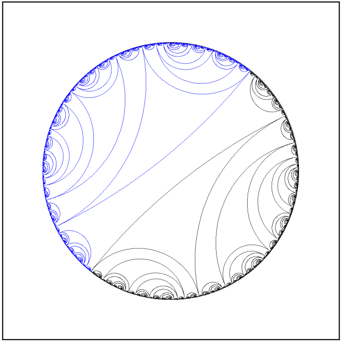

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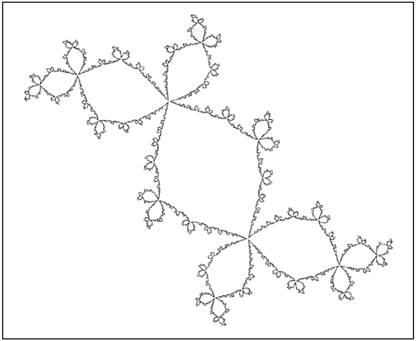

Question. Can we define a transverse measure on $\mathcal{L}_\theta$?

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Let $\theta \in \mathbb{R}/\mathbb{Z}$, and $f_\theta(z) = z^2 + c_\theta$. Then there exists a lamination $\mathcal{L}_\theta$ on the disk such that $\theta_1 \sim \theta_2$ if $R(\theta_1)$ and $R(\theta_2)$ land at the same point.

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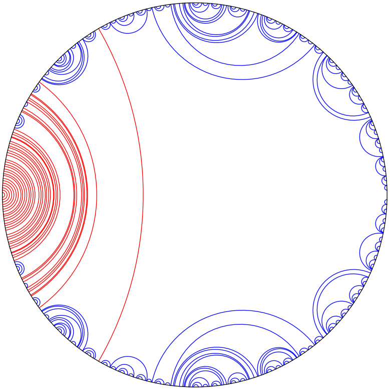
(Compare: Milnor-Thurston, Sullivan dictionary)

Thurston's quadratic minor lamination (QML)



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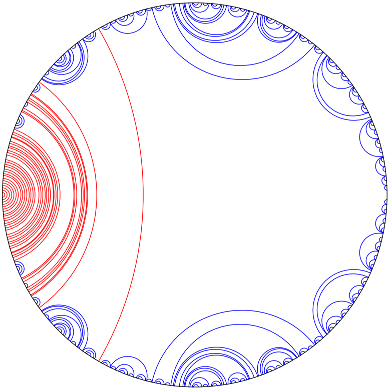

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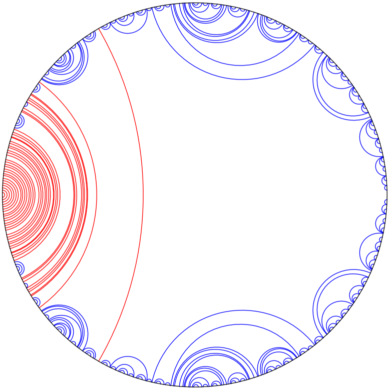

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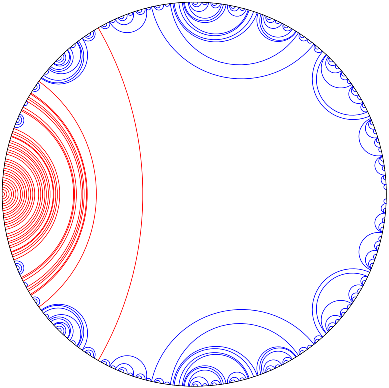
A transverse measure on QML



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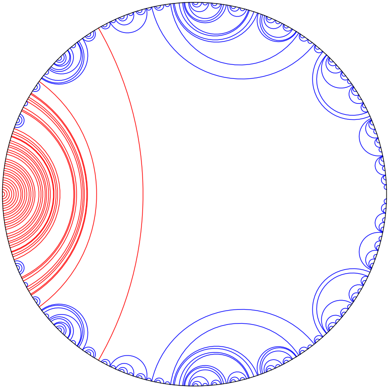

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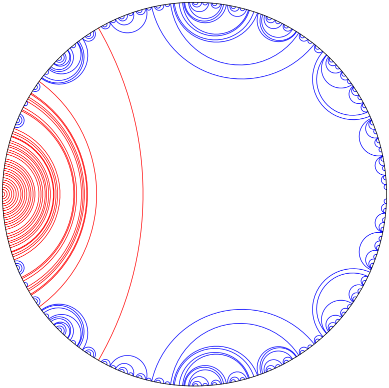

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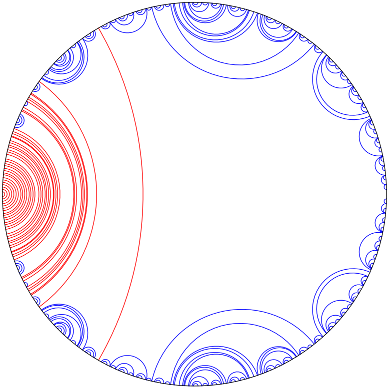

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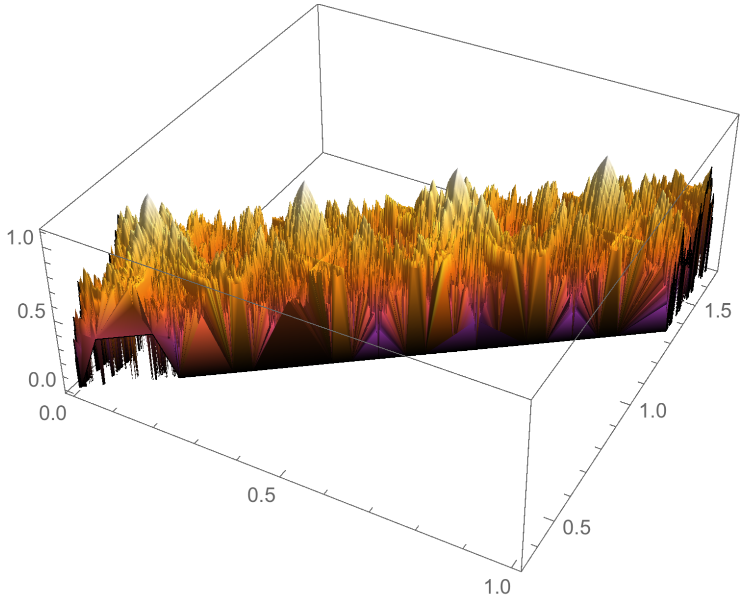
"Combinatorial bifurcation measure"?

The core entropy for cubic polynomials



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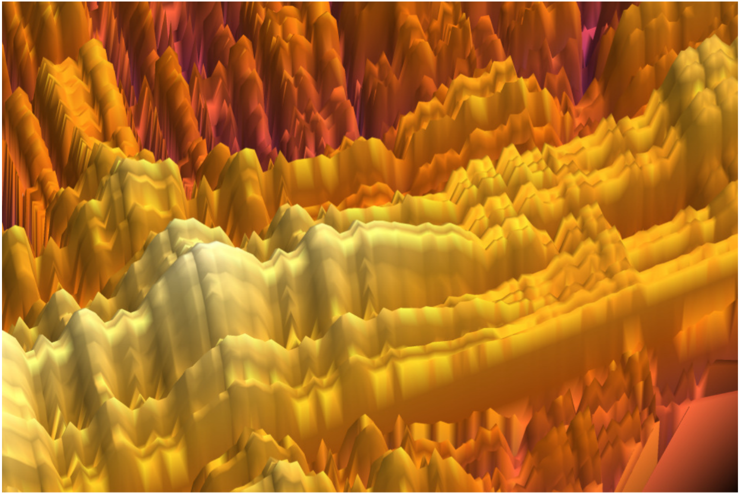

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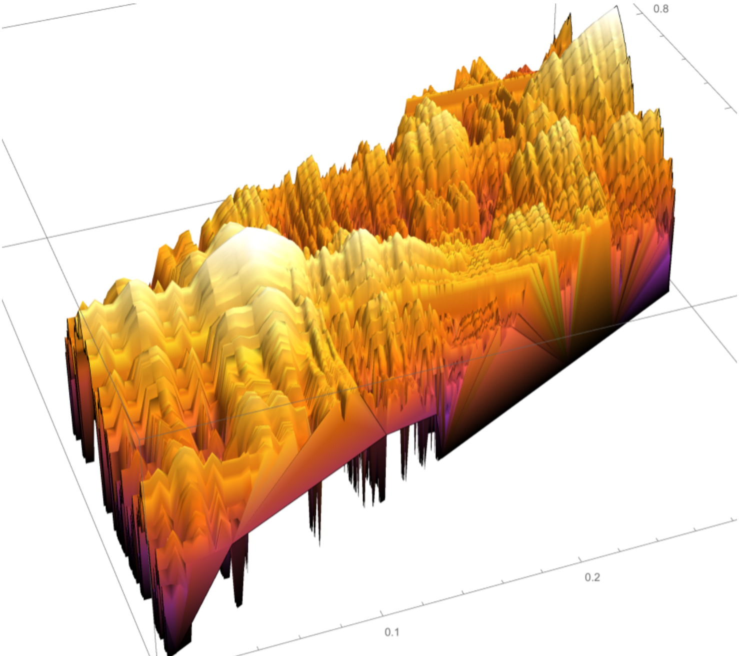
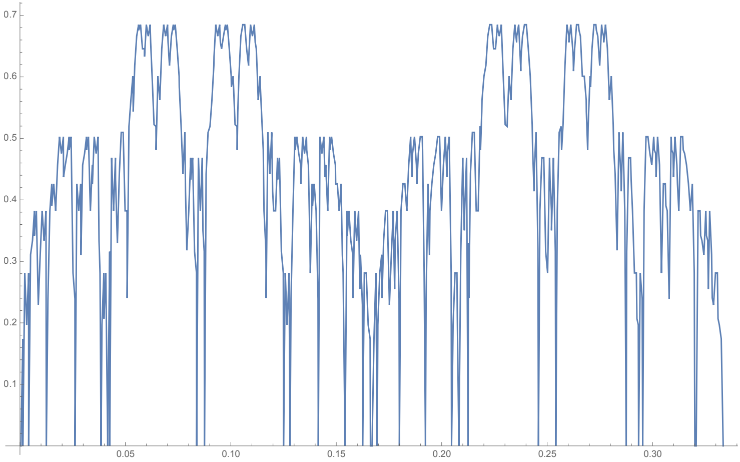


The unicritical slice



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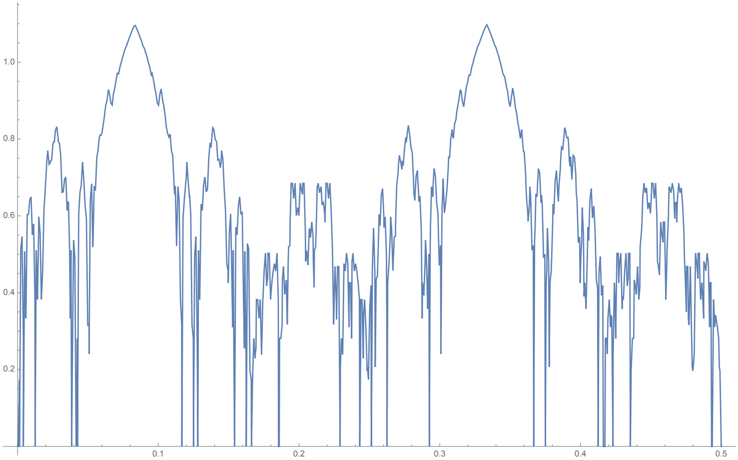

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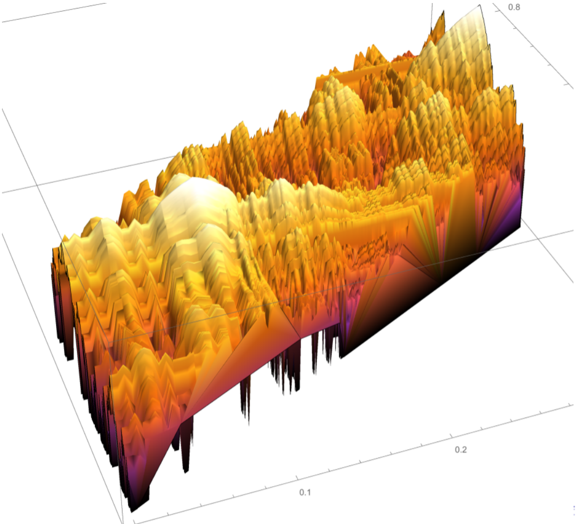

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The end

Thank you!